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STUDY OF THE RANDOM NOISE TEST OF ANALOG-TO-DIGITAL CONVERTERS

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Abstract

An exact expression for the expected value of the mean square difference of the two data sets acquired during the IEEE 1057 Standard Random Noise Test of analog to digital converters is derived. This expression can be used to estimate exactly the amount of random noise present which is an improvement over the heuristically derived estimator suggested in the standard. A study of the influence of stimulus signal amplitude and offset on the existing estimator is carried out.

Keywords: analog-to-digital converter, IEEE 1057, random noise, estimation.

1. Introduction

Knowledge of the amount of random noise in an analog-to-digital converter (ADC) is important for an engineer designing an electronic circuit containing an ADC used to measure a given quantity. This knowledge allows the proper choice of amplification and filtering to use in order to minimize the effects of noise. Knowing the amount of random noise is also very important in ADC testing in two instances: first, when noise itself is used as the stimulus signal [1, 3] and its standard deviation has to be accurately controlled; and second, when it is required to compute the uncertainty of the estimation results, be it in the Histogram Test [4, 5], the Ramp Vernier Test [6], or aperture uncertainty test [7], for instance. In signal processing, where sinusoids are fitted to experimental data points, the value of random noise present can be used to compute the bias [8] and precision [9] of its parameters, like amplitude.

The random noise present in an ADC can be estimated by acquiring two sets of samples and computing the mean square difference between them as suggested in the IEEE 1057-2007 Standard for Digitizing Waveform Recorders [10] and the IEEE 1241-2000 Standard for Terminology and Test Methods for Analog-to-Digital Converters [11]. In both sets the stimulus signal has to be the same for each sample in order for the computed difference to eliminate its influence and leave only the effect of random noise. The natural choice of an input signal is a null voltage. This approach however does not work when the random noise present in the ADC is small when compared with the quantization step since the samples acquired will all have the same value. The computed mean square difference will be zero and consequently a poor estimator of the random noise. In these situations a triangular stimulus signal is used and the acquisition is triggered by it in order for the value of the stimulus signal of each sample to be the same so that it can be eliminated when subtracting the two sets.

Both IEEE standards suggest an empirically derived expression to be used as estimator for the amount of random noise present. This estimator, in certain conditions, may have a large bias, as will be shown later. In [12] the precision of the estimator was studied and an

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analytical expression was derived. Here a new way of estimating the amount of noise present is proposed. Although more complex, it provided more accurate results.

We start by determining the exact expression for the expected value of the mean square difference, given any value of the stimulus signal amplitude, offset and ADC random noise standard deviation (Section 2). This expression will be used to determine the error of the heuristically derived estimator used in the IEEE 1057-2007 Standard (Section 3). The theoretical study carried out will then be used to show how to compute an accurate estimate of the ADC's random noise standard deviation (Section 4). This is a complex procedure which requires the exact knowledge of stimulus signal amplitude and offset and is advantageous when very accurate estimations are required.

2. Theoretical Analysis of the Mean Square Error

Random additive noise in ADCs, as described in [1], is a non-deterministic fluctuation of the ADC output and is described by its frequency spectrum and statistical properties. It is usually considered that the noise present is white (flat frequency spectrum), presenting a stationary probability density function and that the noise is additive and independent of the stimulus signal.

Due to the presence of random noise at the ADC input, the output code (k) can be considered a discrete random variable which can assume any value between 0 and $2^{n_b}-1$ for a n_b-bit ADC. Its statistical properties will depend on the ADC quantization step, Q, and on the amount of random noise, namely, on the value of random noise standard deviation σ_r.

By acquiring two sets, ka and kb, of M samples each and computing the means square difference (or error):

$$mse = \frac{1}{M}\sum_{j=0}^{M-1}\left(ka_j - kb_j\right)^2, \tag{1}$$

it is possible to estimate the standard deviation of the random noise present in the ADC. In the following we will show how to do this.

The two sets of samples acquired are triggered at the same input voltage and thus the value of the stimulus signal for each sample will be the same in both sets. The random noise, however, will be different in each set (ra and rb) which will lead to different output codes. The output codes for both sets, ka ad kb will be given by:

$$\begin{matrix} ka_j = round\left(y_j + ra_j\right) \\ kb_j = round\left(y_j + rb_j\right) \end{matrix} \quad , \quad j = 0...M-1. \tag{2}$$

The output codes can also be defined by:

$$\begin{matrix} ka_j = y_j + ra_j + \nu a_j \\ kb_j = y_j + rb_j + \nu b_j \end{matrix} \quad , \quad j = 0...M-1, \tag{3}$$

where νa_j and νb_j are the quantization errors which are equal to the fractional part of the input voltage y_j:

$$\begin{matrix} \nu a_j = \left\langle y_j + ra_j \right\rangle \\ \nu b_j = \left\langle y_j + rb_j \right\rangle \end{matrix} \quad , \quad j = 0...M-1. \tag{4}$$

Inserting (3) into (1) allows us to express the mean square difference as a function of the random noise and quantization error of the M samples.

$$mse = \frac{1}{M}\sum_{j=0}^{M-1}\left(ra_j - rb_j + \nu a_j - \nu b_j\right)^2. \tag{5}$$

Calculating the square leads to:

$$mse = \frac{1}{M}\sum_{j=0}^{M-1}\left\{\begin{matrix}\left(ra_j^2 + rb_j^2 - 2ra_j rb_j\right) + \left(\nu a_j^2 + \nu b_j^2 - 2\nu a_j \nu b_j\right) + \\ +2\left(ra_j \nu a_j - ra_j \nu b_j - rb_j \nu a_j + rb_j \nu b_j\right)\end{matrix}\right\}. \tag{6}$$

We are interested in determining the expected value of *mse*. To achieve that, we note that the random noise of the two sets is independent and that they have a null mean:

$$\begin{aligned} E\left\{ra_j^2\right\} &= E\left\{rb_j^2\right\} = \sigma_r^2 \\ E\left\{ra_j rb_j\right\} &= E\left\{ra_j\right\}E\left\{rb_j\right\} = 0. \end{aligned} \tag{7}$$

Note that this does not take into account the fact that ADC nonlinearity can influence the statistical distribution of the random noise after quantization.

We can also state that the quantization error of a given sample in one set is independent of the value of the random noise of the same sample in the other set:

$$\begin{aligned} E\left\{ra_j \nu b_j\right\} &= E\left\{ra_j\right\}E\left\{\nu b_j\right\} = 0 \\ E\left\{rb_j \nu a_j\right\} &= E\left\{rb_j\right\}E\left\{\nu a_j\right\} = 0. \end{aligned} \tag{8}$$

Also, since the two sets are statistically identical, we have:

$$E\left\{ra_j \nu a_j\right\} = E\left\{rb_j \nu b_j\right\}. \tag{9}$$

The expected value of *mse* is then:

$$E\{mse\} = \frac{1}{M}\sum_{j=0}^{M-1}\left(2\sigma_r^2 + 2E\left\{\nu a_j^2\right\} - 2E\left\{\nu a_j \nu b_j\right\} + 4E\left\{ra_j \nu a_j\right\}\right). \tag{10}$$

The random noise can be considered stationary and thus all the terms in the summation are equal which allows us to write:

$$E\{mse\} = 2\sigma_r^2 + 2E\left\{\nu a^2\right\} - 2E\{\nu a \cdot \nu b\} + 4E\{ra \cdot \nu a\}. \tag{11}$$

To compute the expected values in (11) we will make use of the characteristic function which is another way of describing a random variable more suitably than the probability density function when we have sums of random variables. By definition, the characteristic function $\Phi_x(u)$ of a random variable x is the expected value of variable e^{jux}. The characteristic function of the quantization error ν is (see [13] (5.10), for a similar situation but without additive noise):

$$\Phi_\nu(u_\nu) = \sum_{l=-\infty}^{\infty}\Phi_y(l\Psi)\cdot\Phi_r(l\Psi)\cdot\Phi_n(u_\nu + l\Psi), \tag{12}$$

where $\Psi = 2\pi$ and n represents noise which is uniformly distributed in the interval [−1/2 ; 1/2]. Note that if the stimulus signal is such that its characteristic function Φ_y in (12) ull for every l different than 0, then we just have $\Phi_\nu(u_\nu) = \Phi_n(u_\nu)$ which means that in such conditions the quantization error can be considered a uniformly distributed random variable [14].

From the characteristic function in (12) is possible to determine the second moment of the quantization error ([13], eq. (3.9)):

$$E\left\{\nu^2\right\} = \frac{1}{j^2} \frac{\partial^2 \Phi_\nu(u_\nu)}{\partial u_\nu^2}\bigg|_{u_\nu=0} . \tag{13}$$

Inserting (12) into (13) leads to:

$$E\left\{\nu^2\right\} = -\sum_{l=-\infty}^{\infty} \Phi_y(l\Psi) \cdot \Phi_r(l\Psi) \cdot \frac{\partial^2 \Phi_n(u_\nu + l\Psi)}{\partial u_\nu^2}\bigg|_{u_\nu=0} . \tag{14}$$

The characteristic function of the uniformly distributed variable n is ([13], eq. (I.82)):

$$\Phi_n(u_n) = E\left\{e^{jnu_n}\right\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{jnu_n} dn = \operatorname{sinc}\left(\frac{u_n}{2}\right). \tag{15}$$

Its second derivative is ([13], eq. (I.84)):

$$\frac{\partial^2 \Phi_n(u_n)}{\partial u_n^2} = \frac{2}{u_n^2}\operatorname{sinc}\left(\frac{u_n}{2}\right) - \frac{2}{u_n^2}\cos\left(\frac{u_n}{2}\right) - \frac{1}{4}\operatorname{sinc}\left(\frac{u_n}{2}\right). \tag{16}$$

Using this we can compute the derivative in (14):

$$\frac{\partial^2 \Phi_n(u_\nu + l\Psi)}{\partial u_\nu^2}\bigg|_{u_\nu=0} = \begin{cases} \dfrac{2}{(l\Psi)^2}\operatorname{sinc}(\pi l) - \dfrac{2}{(l\Psi)^2}\cos(\pi l) - \dfrac{1}{4}\operatorname{sinc}(\pi l) & , \quad l \neq 0 \\ -\dfrac{1}{12} & , \quad l = 0 \end{cases} \tag{17}$$

which can be simplified into:

$$\frac{\partial^2 \Phi_n(u_\nu + l\Psi)}{\partial u_\nu^2}\bigg|_{u_\nu=0} = \begin{cases} -\dfrac{2(-1)^l}{l^2\Psi^2} & , \quad l \neq 0 \\ -\dfrac{1}{12} & , \quad l = 0 \end{cases} \tag{18}$$

The characteristic function of a normally distributed variable, like the random noise here, with null mean and standard deviation σ_r is:

$$\Phi_r(u_r) = E\left\{e^{jru_r}\right\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_r} e^{-\frac{r^2}{2\sigma_r^2}} e^{jru_r} dr = e^{-\frac{\sigma_r^2}{2}u_r^2}. \tag{19}$$

The characteristic function of the stimulus signal depends on the shape of the signal. For a triangular signal with amplitude A_Q and offset C_Q, in LSB, one has:

$$\Phi_y(u_y) = E\left\{e^{jyu_y}\right\} = \int_{C-A_Q}^{C+A_Q} \frac{1}{2A_Q} e^{jyu_y} dy = \operatorname{sinc}(A_Q u_y) e^{jC_Q u_y}. \tag{20}$$

Inserting (18), (19) and (20) into (14), allows the computation of the second moment of the quantization error as a function of stimulus signal amplitude and offset and the random noise standard deviation.

$$E\left\{\nu^2\right\} = \frac{1}{12} + \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q l\Psi\right) e^{jC_Q l\Psi} e^{-\frac{\sigma_r^2}{2}(l\Psi)^2} \frac{2(-1)^l}{l^2\Psi^2}. \tag{21}$$

To compute the third term in the right member of (11) it is necessary to know the joint characteristic function of the quantization error (see [13], eq. (9.4), for a similar situation but without additive noise):

$$\Phi_{\nu a\nu b}\left(u_{\nu a},u_{\nu b}\right) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} \Phi_y\left(l_1\Psi + l_2\Psi\right)\cdot \Phi_{rarb}\left(l_1\Psi, l_2\Psi\right)\cdot \Phi_{nanb}\left(u_{\nu a} + l_1\Psi, u_{\nu b} + l_2\Psi\right). \tag{22}$$

The uniformly distributed random variables *na* and *nb* are independent and their joint characteristic function is just the product of their individual characteristic functions. The same is valid for the random noise *ra* and *rb*. As such we can write (22) as:

$$\Phi_{\nu a\nu b}\left(u_{\nu a},u_{\nu b}\right) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} \Phi_y\left(l_1\Psi + l_2\Psi\right)\cdot e^{\frac{\sigma_r^2}{2}\left(l_1^2+l_2^2\right)\Psi^2}\cdot \Phi_n\left(u_{\nu a} + l_1\Psi\right)\cdot \Phi_n\left(u_{\nu b} + l_2\Psi\right). \tag{23}$$

The expected value of the product of νa by νb can now be computed from this characteristic function.

$$E\{\nu a\nu b\} = \frac{1}{j^2}\frac{\partial^2\Phi_{\nu a\nu b}\left(u_{\nu a},u_{\nu b}\right)}{\partial u_{\nu a}u_{\nu b}}\Bigg|_{\substack{u_{\nu a}=0\\ u_{\nu b}=0}}. \tag{24}$$

Inserting (23) leads to:

$$E\{\nu a\nu b\} = -\sum_{\substack{l_1=-\infty\\ l_1\neq 0}}^{\infty}\sum_{\substack{l_2=-\infty\\ l_2\neq 0}}^{\infty} \Phi_y\left(l_1\Psi + l_2\Psi\right)\cdot e^{-\frac{\sigma_r^2}{2}\left(l_1^2+l_2^2\right)\Psi^2}\cdot\frac{(-1)^{l_1+l_2}}{l_1 l_2\Psi^2}, \tag{25}$$

where we have used the fact that (see [13], eq. (I.83)):

$$\frac{\partial\Phi_n\left(u_n + l\Psi\right)}{\partial u_n}\Bigg|_{u_n=0} = \begin{cases} \left[\dfrac{\cos\left(\dfrac{u_n + l\Psi}{2}\right)}{u_n + l\Psi} - \dfrac{\operatorname{sinc}\left(\dfrac{u_n + l\Psi}{2}\right)}{u_n + l\Psi}\right]_{u_n=0} = \dfrac{(-1)^l}{l\Psi} & , \ l\neq 0 \\ 0 & , \ l = 0 \end{cases} \tag{26}$$

Using (20) we can write:

$$E\{\nu a\nu b\} = -\sum_{\substack{l_1=-\infty\\ l_1\neq 0}}^{\infty}\sum_{\substack{l_2=-\infty\\ l_2\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q\left(l_1\Psi + l_2\Psi\right)\right) e^{jC_Q\left(l_1\Psi + l_2\Psi\right)}\cdot e^{-\frac{\sigma_r^2}{2}\left(l_1^2+l_2^2\right)\Psi^2}\cdot\frac{(-1)^{l_1+l_2}}{l_1 l_2\Psi^2}. \tag{27}$$

The forth term in (11) is computed from the joint characteristic function of the quantization error and random noise.

$$\Phi_{r\nu}\left(u_r,u_\nu\right) = \sum_{l=-\infty}^{\infty} \Phi_y\left(l\Psi\right)\cdot\Phi_r\left(u_r + l\Psi\right)\cdot\Phi_n\left(u_\nu + l\Psi\right). \tag{28}$$

The expected value of $r\nu$ is:

$$E\{r\nu\} = \frac{1}{j^2}\frac{\partial^2\Phi_{r\nu}\left(u_r,u_\nu\right)}{\partial u_r u_\nu}\Bigg|_{\substack{u_r=0\\ u_\nu=0}}. \tag{29}$$

Inserting (28) leads to:

$$E\{r\nu\} = -\sum_{l=-\infty}^{\infty} \Phi_y(l\Psi) \cdot \left.\frac{\partial \Phi_r(u_r + l\Psi)}{\partial u_r}\right|_{u_r=0} \cdot \left.\frac{\partial \Phi_n(u_\nu + l\Psi)}{\partial u_\nu}\right|_{u_\nu=0} . \tag{30}$$

Using (19) one has:

$$\frac{\partial \Phi_r(u_r)}{\partial u_r} = -u_r \sigma_r^2 e^{-\frac{\sigma_r^2}{2} u_r^2} . \tag{31}$$

Inserting (31) and (26) into (30) leads to:

$$E\{r\nu\} = \sum_{l=-\infty}^{\infty} \Phi_y(l\Psi) \cdot \sigma_r^2 e^{-\frac{\sigma_r^2}{2} l^2 \Psi^2} \cdot (-1)^l . \tag{32}$$

Using (20) we can write:

$$E\{r\nu\} = \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q l\Psi\right) e^{jC_Q l\Psi} \cdot \sigma_r^2 e^{-\frac{\sigma_r^2}{2} l^2 \Psi^2} \cdot (-1)^l . \tag{33}$$

It is now possible to exactly compute the expected value of the mean square error by inserting (21), (27) and (33) into (11):

$$\begin{aligned} E\{mse\} = {} & 2\sigma_r^2 + \frac{1}{6} + 4 \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q l\Psi\right) e^{jC_Q l\Psi} e^{-\frac{\sigma_r^2}{2} l^2 \Psi^2} \frac{(-1)^l}{l^2 \Psi^2} + \\ & + 2 \sum_{\substack{l_1=-\infty \\ l_1\neq 0}}^{\infty} \sum_{\substack{l_2=-\infty \\ l_2\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q (l_1\Psi + l_2\Psi)\right) e^{jC_Q (l_1\Psi + l_2\Psi)} \cdot e^{-\frac{\sigma_r^2}{2}\left(l_1^2 + l_2^2\right)\Psi^2} \cdot \frac{(-1)^{l_1+l_2}}{l_1 l_2 \Psi^2} + \\ & + 4 \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q l\Psi\right) e^{jC_Q l\Psi} \cdot \sigma_r^2 e^{-\frac{\sigma_r^2}{2} l^2 \Psi^2} \cdot (-1)^l . \end{aligned} \tag{34}$$

Note that the complex exponential of $C_Q l\Psi$ is the same as $\cos\left(C_Q l\Psi\right) + j\sin\left(C_Q l\Psi\right)$ which is the sum of an even and an odd part. In the summations the odd part cancels out leaving only its even part. We can thus write (34) as:

$$\begin{aligned} E\{mse\} = {} & 2\sigma_r^2 + \frac{1}{6} + 8 \sum_{l=1}^{\infty} \operatorname{sinc}\left(A_Q l\Psi\right) \cos\left(C_Q l\Psi\right) e^{-\frac{\sigma_r^2}{2} l^2 \Psi^2} \frac{(-1)^l}{l^2 \Psi^2} + \\ & + 4 \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \operatorname{sinc}\left(A_Q (l_1\Psi + l_2\Psi)\right) \cos\left(C_Q (l_1\Psi + l_2\Psi)\right) \cdot e^{-\frac{\sigma_r^2}{2}\left(l_1^2 + l_2^2\right)\Psi^2} \cdot \frac{(-1)^{l_1+l_2}}{l_1 l_2 \Psi^2} + \\ & + 4 \sum_{l_1=-\infty}^{-1} \sum_{l_2=1}^{\infty} \operatorname{sinc}\left(A_Q (l_1\Psi + l_2\Psi)\right) \cos\left(C_Q (l_1\Psi + l_2\Psi)\right) \cdot e^{-\frac{\sigma_r^2}{2}\left(l_1^2 + l_2^2\right)\Psi^2} \cdot \frac{(-1)^{l_1+l_2}}{l_1 l_2 \Psi^2} + \\ & + 8 \sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \operatorname{sinc}\left(A_Q l\Psi\right) \cos\left(C_Q l\Psi\right) \cdot \sigma_r^2 e^{-\frac{\sigma_r^2}{2} l^2 \Psi^2} \cdot (-1)^l . \end{aligned} \tag{35}$$

In Fig. 1 we can see this expected value as a function of stimulus signal amplitude and offset for three different values of random noise standard deviation. The oscillations of the expected value with amplitude and offset are due to the “sinc” function of A_Q and the cosine function of C_Q in the characteristic function of the stimulus signal y in (20).

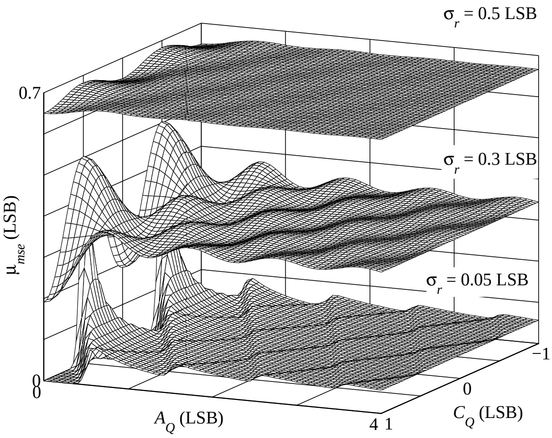

Fig. 1. Representation of the expected value of the mean square error for a triangular stimulus signal as a function of its amplitude (A_Q) and offset (C_Q) for three values of random noise standard deviation.

From Fig. 1 it is also clear that those oscillations decrease with the stimulus signal amplitude. This is because the characteristic function of y has the term $\mathrm{sinc}\left(A_Q u_y\right)$ which decreases with stimulus signal amplitude. The oscillations of the expected value of *mse* also decrease with the random noise standard deviation. This comes from the negative exponential term of σ_r^2 in eq. (21), eq. (27) and eq. (33). As the random noise standard deviation increases, the exponential term tends to 0 and all the summations become null. The expected value of *mse* becomes, from (11):

$$E\{mse\} = 2\left(\sigma_r^2 + \frac{1}{12}\right), \tag{36}$$

which is just double the variance of the random noise plus the variance of the quantization error (1/12). It is double because of the two sets of samples that are subtracted in the computation of *mse*. Expression (36) is expected since for high values of random noise standard deviation we can consider the random noise to be statistically independent of the quantization error. Therefore we just have the variance of the sum of two independent random variables which is the sum of their respective variances.

The value assumed by the expected value of *mse* for an integer A_Q is the same value it assumes when A_Q tends to ∞ as seen by the dashed line in Fig. 2.

This can also be seen in Fig. 3 where we have plotted the expected value of the mean square error as a function of random noise standard deviation for different values of stimulus signal amplitude ranging from 0 to 1 LSB. Note that for higher values of A_Q the range covered is the same as seen in Fig. 2. For values of σ_r^2 greater than 0.5 LSB we see that the expected value of *mse* does not change significantly with A_Q.

The solid line in Fig. 3 represents the case when the stimulus signal amplitude is a multiple of the quantization step of the ADC (A_Q integer).

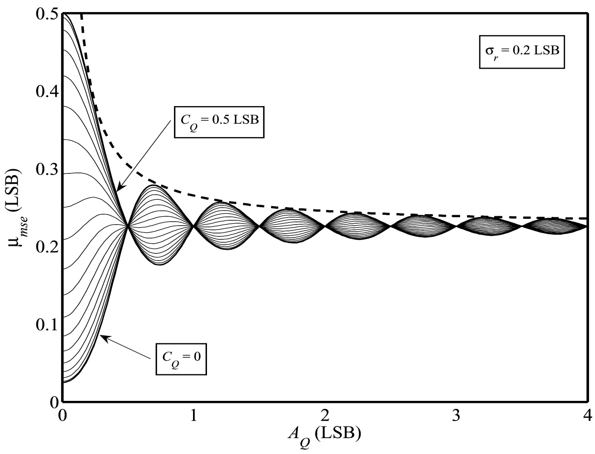

Fig. 2. Representation of the expected value of the mean square error as a function of stimulus signal amplitude (A_Q). The dashed line is an approximation to the envelope of the expected value.

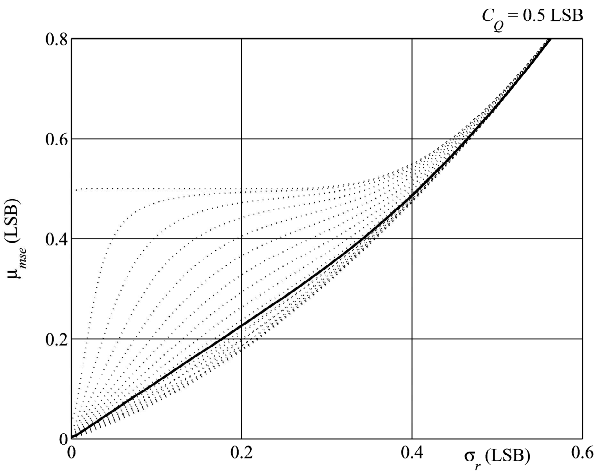

Fig. 3. Representation of the expected value of the mean square error as a function of the random noise standard deviation for different values of triangular stimulus signal amplitude ranging from 0 to 1 LSB (dotted lines). The solid line represents the case where A_Q = 1 LSB.

3. Error of the IEEE 1057-2007 Standard estimator

Having determined an exact expression for the expected value of the mean square difference as a function of stimulus signal amplitude, offset and the ADCs' random noise standard deviation, it is now possible to compute the expected value of the estimator used in the IEEE 1057-2007 standard [10]:

$$\sigma_r = \frac{1}{\sqrt[4]{\dfrac{1}{\left(\sqrt{\dfrac{mse}{2}}\right)^4} + \dfrac{1}{\left(\dfrac{\sqrt{\pi}}{2} mse\right)^4}}}, \tag{37}$$

The expected value of this estimator can be approximated by:

$$\mu_{\sigma_r} \approx \frac{1}{\sqrt[4]{\dfrac{1}{\left(\sqrt{\dfrac{\mu_{mse}}{2}}\right)^4} + \dfrac{1}{\left(\dfrac{\sqrt{\pi}}{2} \mu_{mse}\right)^4}}}. \tag{38}$$

We now define the error of the estimator as the difference between its expected value and the actual value of the standard deviation of random noise present:

$$e_{\sigma_r} = \mu_{\sigma_r} - \sigma_r. \tag{39}$$

Fig. 4 represents the estimation error of the random noise standard deviation as a function of the actual random noise standard deviation for different values of triangular stimulus signal amplitude ranging from 1 to 2 LSB (thin lines). The thick line represents the maximum value approximation valid for $A_Q > 1$ LSB.

In Fig. 5 a more limited range of values of noise standard deviation is shown (from 0 to 1 LSB). It can be seen, for example, that for a random noise standard deviation of 0.2 LSB the estimation error could be greater than 0.03 LSB which is a relative error higher than 15% (for values of stimulus signal amplitude greater than 1 LSB).

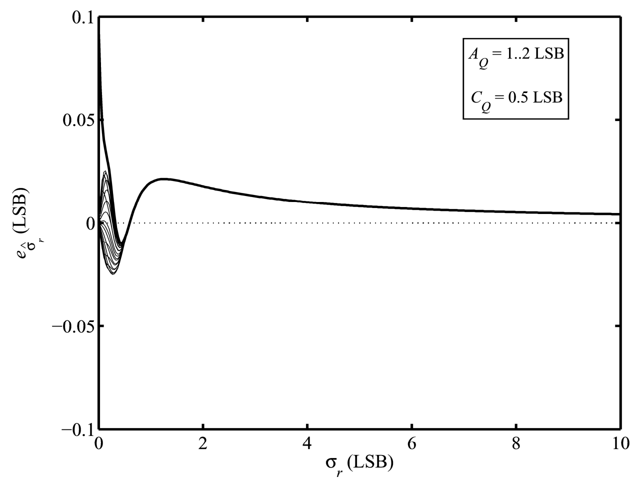

Fig. 4. Representation of the estimation error of the random noise standard deviation as a function of the actual random noise standard deviation for different values of stimulus signal amplitude ranging from 1 to 2 LSB (thin lines). The thick line represents the maximum value approximation valid for $A_Q > 1$ LSB.

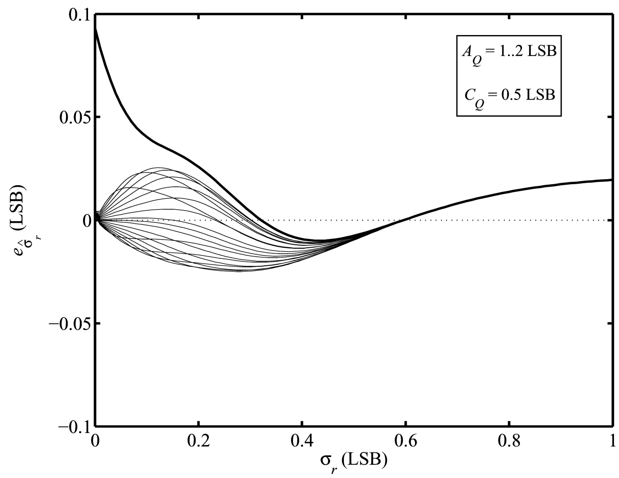

Fig. 5. Representation of the estimation error of the random noise standard deviation as a function of the actual random noise standard deviation for different values of stimulus signal amplitude ranging from 1 to 2 LSB (thin lines). The thick line represents the maximum value approximation valid for $A_Q > 1$ LSB.

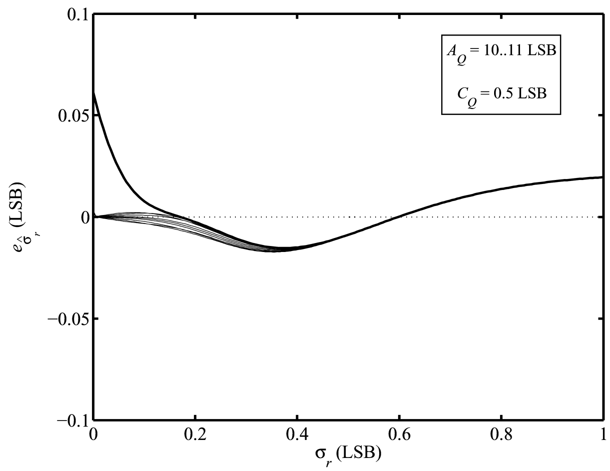

Fig. 6. Representation of the estimation error of the random noise standard deviation as a function of the actual random noise standard deviation for different values of triangular stimulus signal amplitude ranging from 10 to 11 LSB (thin lines). The thick line represents the maximum value approximation valid for $A_Q > 1$ LSB.

In Fig. 6 a case for higher values of stimulus signal amplitude is depicted showing the smaller estimation error in these circumstances.

4. Estimating Exactly the Amount of Random Noise

As seen in the previous section, the expected value of the mean square error can be computed using (35) for a given value of random noise standard deviation. This will give an unbiased estimation of the random noise if the stimulus signal's amplitude and offset are known. Since there is a one-to-one relation between the expected value of the mean square error and the random noise it is possible to use the measured mean square error and obtain an estimate of the random noise standard deviation by solving (35) for σ_r^2. This has to be done numerically since there is no analytical expression for it. This can be done with a computer, limiting the computation of the summations in (35) to a finite number of terms. This is possible, since the terms of the summations become smaller as the index variables (l, l_1 and l_2) grow. In Fig. 7 we show the estimation error for three different values of P where $+P$ and $-P$ are the limits of the summations. As we can see, a value of 30 is good enough for all practical situations. The computational burden of numerically solving (35) with 30 terms in each summation is negligible for modern computers (less than 100 ms).

Note that the estimation error decreases with increasing random noise standard deviation since the terms in the summations depend on $e^{-\sigma_r^2}$. For values of σ_r greater than 0.2 LSB even $P = 1$ is enough.

Using only one term in the summations in (35) leads to a simpler expression:

$$E\{mse\} = 2\sigma_r^2 + \frac{1}{6} - 8\operatorname{sinc}\left(A_Q\Psi\right)\cos\left(C_Q\Psi\right)e^{-\frac{\sigma_r^2}{2}\Psi^2}\frac{1}{\Psi^2} + \\ +4\operatorname{sinc}\left(2A_Q\Psi\right)\cos\left(2C_Q\Psi\right)e^{-\sigma_r^2\Psi^2}\frac{1}{\Psi^2} - 4e^{-\sigma_r^2\Psi^2}\frac{1}{\Psi^2} - 8\operatorname{sinc}\left(A_Q\Psi\right)\cos\left(C_Q\Psi\right)\sigma_r^2 e^{-\frac{\sigma_r^2}{2}\Psi^2}. \tag{40}$$

Looking at Fig. 7 one can see that for values of noise standard deviation higher than 0.1 LSB the error incurred in using (40) instead of (35) is lower than 0.005 LSB.

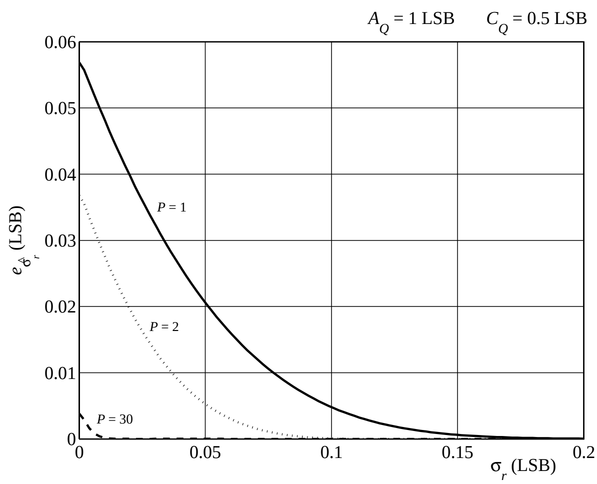

Fig. 7. Representation of the error in estimate of the random noise standard deviation due to the computation of a finite number of terms (P) in the infinite summations of eq. (21), (27) and (33).

5. Conclusions

In this paper an exact expression for the expected value of the mean square difference of the two data sets acquired in the 1057-2007 Standard Random Noise Test of ADC was derived (expression (35)). This expression was used to show the bias of the IEEE 1057-2007 Standard heuristically derived estimator and to propose a new way of estimating the amount of random noise present which gives unbiased results. The increased computational burden may be justified when accuracy is paramount.

The advantage of using the expression presented here, in terms of accuracy, is most significant when the amount of random noise is smaller than 1 LSB. In practice this situation is encountered in fast, low resolution ADCs used, for instance, in telecommunications.

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